Conformal Field Theory and Gravity

Solutions to Problem Set 2 Fall 2024

1. Mechanics of spherically symmetric black holes

(a) Neglecting, the S^2 factor, the geometry near the horizon is

$$ds^2 = f'(r_h)(r - r_h)d\tau^2 + \frac{dr^2}{f'(r_h)(r - r_h)} \tag{1}$$

Using a transformation $r - r_h = \rho^2$, we get

$$ds^2 = \frac{4}{f'(r_h)}\Big(\frac{(f'(r_h))^2}{4}\rho^2 d\tau^2 + d\rho^2\Big) \tag{2}$$

This is regular at $\rho = 0$ if the manifold is locally isomorphic to $\mathbb{R}^2$, hence we require $\tau \sim \tau + \frac{4\pi}{|f'(r_h)|}$.

(b) The relation between surface gravity and temperature is fixed by black hole thermodynamics $\kappa = 2\pi T_h$, and the interpretation of the area of a black hole as its entropy $S = \frac{A}{4}$. One matches the first laws

$$\frac{\kappa}{8\pi}dA = dM + ... \tag{3}$$

$$TdS = dE + ... \tag{4}$$

Let us find now the surface gravity with a geometric approach. In EF coordinates we get

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{5}$$

If the metric is asymptotically flat, then the vector $\xi = \partial_u$ satisfies our requirements: $\xi^2 = -f(r) \to -1$ as $r \to \infty$ and it is a null Killing vector at $r = r_h$. Therefore, we get

$$\nabla_\mu(\xi^2)|_{r_h} = -f'(r_h)\delta^r_\mu \tag{6}$$

Using $\xi_\mu|_{r_h} = g_{\mu v}|_{r_h} = \delta^r_\mu$, we obtain the surface gravity

$$\kappa = \frac{|f'(r_h)|}{2} \tag{7}$$

(c) The black hole has an horizon at $r_\pm = M \pm \sqrt{M^2 - Q^2}$, with $r_h \equiv r_+$ being the outer one.
If we write $f(r) = \frac{(r-r_-)(r-r_+)}{r^2}$, we easily find the temperature is

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - Q^2}}{2\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2})} \tag{8}$$

For Schwarzschild, we recover $T = \frac{1}{8\pi M}$.

(d) The area of the black hole is simply

$$A = 4\pi r_+^2 = 4\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}) \tag{9}$$

For Schwarzschild, we recover $A = 16\pi M^2$.

(e) The electric potential is given by:

$$\Phi = -\frac{T}{4}\frac{\partial A}{\partial Q} \tag{10}$$

(f) Before the collision, the total area of the two black holes is $A_i = 16\pi(M_1^2 + M_2^2)$. After the collision, it is $A_f = 16\pi M_3^2$.
The second law implies $A_f \geq A_i$, hence $M_3 \geq M_{cr} = \sqrt{M_1^2 + M_2^2}$.

2. Raychaudhuri equation

(a) Since we assume $\hat{\omega}_{\mu\nu} = 0$ and since $\hat{\sigma}_{\mu\nu}$ is spacelike, we have that

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu \leq -\frac{1}{2}\theta^2 - R_{\mu\nu}k^\mu k^\nu \tag{11}$$

Using the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{12}$$

One can contract with $g^{\mu\nu}$ to obtain

$$R = -8\pi G T \tag{13}$$

where $T \equiv T_{\mu\nu}g^{\mu\nu}$. Plugging this back into the original Einstein's equation, we get

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) \tag{14}$$

Now contracting with $k^\mu k^\nu$ using that $k^\mu k_\mu = 0$ and the null energy condition, we obtain

$$R_{\mu\nu}k^\mu k^\nu = 8\pi G T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{15}$$

Thus,

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \tag{16}$$

Rearranging,

$$d\left(\frac{1}{\theta}\right) \geq \frac{1}{2}d\lambda \tag{17}$$

Integrating between $(0, \theta_0)$ and (λ_0, θ_f), we get

$$\frac{1}{\theta_f} - \frac{1}{\theta_0} \geq \frac{\lambda_0}{2} \tag{18}$$

taking $\theta_f \to \infty$ and θ_0 negative, we obtain the suggested bound,

$$\lambda_0 \leq \frac{2}{|\theta_0|} \tag{19}$$

(b) As the hint suggests, we need to verify $\xi^\alpha \nabla_\alpha \xi^\mu = 0$. Since ξ^μ only has a V component, this reduces to

$$0 = \xi^V \nabla_V \xi^\mu = \xi^V \partial_V \xi^\mu + \Gamma^\mu_{VV} \xi^V \tag{20}$$

Using

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\nabla_\mu g_{\nu\sigma} + \nabla_\nu g_{\mu\sigma} - \nabla_\sigma g_{\mu\nu}) \,, \tag{21}$$

we get that that the only non-trivial component of Γ^μ_{VV} is

$$\Gamma^V_{VV} = g^{-1}_{UV} \partial_V (g_{UV}) \tag{22}$$

where g_{UV} is the component of the metric

$$g_{UV} = -\frac{16M^3 e^{-r/(2M)}}{r} \tag{23}$$

To compute $\partial_V \xi^V$, we need $\partial_V r$. This is computed by deriving the implicit relation $r = r(U, V)$ on both sides with respect to V. We obtain

$$\partial_V r = -U \frac{4M^2}{r} e^{-r/(2M)} \tag{24}$$

Plugging everything and computing the derivatives, we obtain the desired result.

(c) Using the hint,

$$\theta = r^{-1} e^{r/(2M)} \partial_V (\underbrace{r e^{-r/(2M)}}_{\sqrt{-g}} r e^{r/(2M)}) = 2 e^{r/(2M)} \partial_V r \tag{25}$$

Using the result obtained previously for $\partial_V r$, we get

$$\theta = -\frac{8M^2}{r} U \tag{26}$$

(d) Using the chain rule,

$$\frac{d\theta}{d\lambda} = \frac{dV}{d\lambda} \frac{d\theta}{dV} \tag{27}$$

and using $dV/d\lambda = \xi^V = r e^{r/(2M)}$ by definition of ξ^μ, taking the derivative $d\theta/dV$ using the expression of θ and the formula for $\partial_V r$, we obtain

$$\frac{d\theta}{d\lambda} = \underbrace{-\frac{32M^4}{r^3} U e^{-r/(2M)}}_{d\theta/dV} r e^{r/(2M)} = -\frac{1}{2}\theta^2 \tag{28}$$

(e) Follow the discussion in point (a) by replacing $\leq$, $\geq$ by $=$. Our situation thus corresponds to the upper bound $\lambda_0 = 2/|\theta_0|$.